The anyon model: an example inspired by string theory

S.V. Talalov

Department of Theoretical Physics, State University of Tolyatti,
14 Belorusskaya str., Tolyatti, Samara region, 445667 Russia.
svtalalov@tltsu.ru

Abstract

We investigate the enlarged class of open finite strings in $(2+1)D$ space-time. The new dynamical system related to this class is constructed and quantized here. As the result, the energy spectrum of the model is defined by a simple formula $\mathsf{S} = \alpha_n \mathsf{E} + c_n$; the spin S is an arbitrary number here but the constants α_n and c_n are eigenvalues for certain spectral problems in fermionic Fock space $\mathbf{H}_\psi$ constructed for the free $2D$ fermionic field.

keywords: anyon models, noncritical strings, boson-fermion correspondense.

PACS numbers: 11.25.Pm, 04.60.Kz, 02.40.Hw.

1 Introducton

As it seems, "anyons"[1] – the particles with arbitrary spin and statistics – can be realised as the exitations of some infinity-dimensional dynamical system on a plane[2]. The finite planar string is the simplest example of this system. Note that a non-standard point of view on elementary particles was suggested recently[3]. It described them as the defects of the string condensed matter ("string-net condensation"). Open string in arbitrary space-time dimensions is a well-investigated object (see, e.g. [4, 5]). One of the frequently discussed structures here is the first form **I** of the world-sheet – as opposed to the second form **II**. In this work we investigate the finite string on a spatial plane in terms of second form **II** and construct the certain dynamical system related to this string. We interprete the excitations of constructed sysytem as anyon-type quasiparticles.

Let us begin with the classical theory. The suggested scheme [6] generalizes the standard geometrical approach [7] in string theory. Here we will briefly remind the main points of our approach. We start with the Nambu-Goto action

$$S = -\gamma \int \sqrt{-\det(\partial_i \mathbf{X} \partial_j \mathbf{X})}\, d\xi^0 d\xi^1\,, \tag{1}$$

where $\mathbf{X} = \mathbf{X}(\xi^0, \xi^1)$ is the vector in Minkowski space-time $E_{1,3}$, the parameters ξ^0 and ξ^1 are world-sheet parameters and symbols ∂_i denote corresponding derivatives. Thus we will consider the minimal surfaces in the space-time $E_{1,3}$. It is well-known (see, for example [4, 7, 8]) that the special parametrization of the world-sheet can be selected – so that both equalities

$$\partial_+ \partial_- \mathbf{X} = 0\,, \tag{2}$$

the constraints

$$(\partial_\pm \mathbf{X})^2 = 0 \tag{3}$$

and the boundary conditions

$$\partial_1 \mathbf{X}\Big|_{\xi^1=0} \quad = \quad \partial_1 \mathbf{X}\Big|_{\xi^1=\pi} \quad = \quad 0 \tag{4}$$

will be fulfilled. We denote $\partial_\pm = \partial/\partial\xi_\pm$ and $\xi_\pm = \xi^1 \pm \xi^0$ here. Thus the initial objects of our investigation are the time-like world-sheets with orthonormal parametrization. Our initial steps in sections 1 and 2 will be:

- We will reduce our theory to $3D$ case;
- We will define new bijective parametrization for the world-sheet variables $\mathbf{X}(\xi^0, \xi^1)$ constrained by the equalities (3) so that

  $$\mathbf{X}(\xi^0, \xi^1) = \mathbf{X}\left[\varkappa;\, \mathbf{Z},\, B;\, \varphi(\xi^0, \xi^1),\, \alpha_+(\xi^0, \xi^1),\, \alpha_-(\xi^0, \xi^1)\right], \tag{5}$$

  where the new (unconstrained) parameters will be transformed differently for the scale and Poincaré transformations of space-time. So, the Poincaré transformations of the world-sheet will transform the constant vector $\mathbf{Z} \in E_{1,2}$ and the constant matrix $B \in SL(2, R)$ only; the scale transformations will transform the real constant $\varkappa \in (0, \infty)$ only. The functions φ, $\alpha_\pm$ will be certain relativistic and scale invariant functions.
- We will factorize the set $\mathcal{X}$ of the considered world-sheets on the orbits for certain gauge group G_0 .

In section 3 we consider the set $\mathcal{X}/\mathsf{G}_0$ as a dynamical system; the ξ^0-dynamics is defined here by the differential equations (2) and the conditions (3) and (4). The well-known fact that same dynamical system can have the different hamiltonian structures (see[9], for example). In accordance with the Dirac ideas[10] we define the hamiltonian structure for our dynamical system as initial conception. The constructed phase space $\mathcal{H}$ will be costrained

by the finite number of constraints. What are the reasons to consider the constructed theory to be string related theory? Let the set $\mathcal{V}_{str} \subset \mathcal{H}_{str}$ be the surface of first type constraints (3) in the standard string phase space $\mathcal{H}_{str}$ and the set $\mathcal{V} \subset \mathcal{H}$ – first type constraint surface in the constructed phase space $\mathcal{H}$. We will have the following one-to-one correspondence $\leftrightarrow$:

$$(\mathcal{V}_{str}/\mathsf{G}_0) \longleftrightarrow \mathcal{V} \subset \mathcal{H}\,.$$

Moreover the correspondence $\leftrightarrow$ will be constructed so that the momentum and the angular moment for the defined dynamical system will be equal to the Nöether momentum and the angular moment for string. Thus both physical degrees of freedom and the dynamical invariants will coincide for the string and the constructed dynamical system on the classical level. This fact makes it possible to interpret the constructed dynamical system as some finite extended object on a plane. We emphasize that there is no canonical transformation which connects the phase space $\mathcal{H}_{str}$ and the phase space $\mathcal{H}$. As the result, the quantum theory (that is constructed in section 4 with the help of boson-fermion correspondence method) differs from the standard quantum theory for strings.

Let us execute the programm outlined above. Firstly we define a pair of light-like and scale-invariant vectors in space $E_{1,3}$:

$$\mathbf{e}_\pm(\xi_\pm) = \pm\frac{1}{\varkappa}\,\partial_\pm \mathbf{X}(\xi_\pm)\,, \tag{6}$$

where $\varkappa$ is an arbitrary positive constant. If the vectors $\mathbf{X} \in E_{1,3}$ are transformed as $\mathbf{X} \to \widetilde{\mathbf{X}} = \lambda\mathbf{X}$, the constant $\varkappa$ is transformed as $\varkappa \to \widetilde{\varkappa} = \lambda\varkappa$. Thus we separate out the scale-transformed mode by the introduction of the variable $\varkappa$ and the projective vectors $\mathbf{e}_\pm$. As the action (1) describes the scale-invariant theory, we consider this step to be justified here.

Secondly we define a pair of orthonormal bases [11] $\mathbf{e}_{\nu\pm}(\xi_\pm)$ that satisfy the conditions $\mathbf{e}_\pm = (\mathbf{e_{0\pm}} \mp \mathbf{e_{3\pm}})\,/2$. Instead of vectors $\mathbf{e}_{\nu\pm}$ we can consider the vector-matrices $\hat{\mathbf{E}}_\pm$:

$$\hat{\mathbf{E}}_\pm = \mathbf{e_{0\pm}}\mathbf{1_2} - \sum_{i=1}^{3} \mathbf{e_{i\pm}}\boldsymbol{\sigma_i}\,, \tag{7}$$

these matrices are more convenient here. We require that all the other elements of the matrix $\hat{\mathbf{E}}_+$ ($\hat{\mathbf{E}}_-$)depend on the variable ξ_+ (ξ_-) only just as the vector $\mathbf{e}_+$ ($\mathbf{e}_-$) does. It is clear that the definition of the bases $\mathbf{e}_{\nu\pm}(\xi_\pm)$ has three - parameter arbitrariness in each point (ξ^0, ξ^1); we intend to return to

this question later. The principal object of our approach is the $SL(2, C)$ - valued field $K(\xi^0, \xi^1)$ which is defined as follows:

$$\hat{\mathbf{E}}_+ = K\hat{\mathbf{E}}_- K^+ \,. \tag{8}$$

To make the reduction to D = 1+2 space-time we require that matrix $K \in SL(2, R)$. This requirement means that

$$\mathbf{e_2}_+(\xi_+) = \mathbf{e_2}_-(\xi_-) = \mathbf{b}_2 \,,$$

where $\mathbf{b}_2$ is a constant spatial vector. Thus the reduced space-time is any space $E_{1,2} \perp \mathbf{b}_2$. All these spaces are equivalent here.

In accordance with the definition, the matrix field $K(\xi^0, \xi^1)$ satisfies to (special) WZWN - equation

$$\partial_+ \left(K^{-1}\partial_- K\right) = 0 \,. \tag{9}$$

Let us define the real functions $\varphi(\xi^0, \xi^1)$ and $\alpha_\pm(\xi^0, \xi^1)$ by means of Gauss decomposition for the matrix $K(\xi^0, \xi^1)$:

$$K = \begin{pmatrix} 1 & 0 \\ -\alpha_+ & 1 \end{pmatrix} \begin{pmatrix} \exp(-\varphi/2) & 0 \\ 0 & \exp(\varphi/2) \end{pmatrix} \begin{pmatrix} 1 & \alpha_- \\ 0 & 1 \end{pmatrix} . \tag{10}$$

In general, these functions are singular because the decomposition (10) is not defined for the points where the principal minor K_{11} vanishes. Let us introduce regular functions $\rho_\pm = (\partial_\pm \alpha_\mp) \exp(-\varphi)$. As the consequence of the equality (9)we will get the following PDE - system:

$$\partial_+\partial_-\varphi = 2\rho_+\rho_- \exp\varphi, \tag{11a}$$

$$\partial_\pm \rho_\mp = 0, \tag{11b}$$

$$\partial_\pm \alpha_\mp = \rho_\pm \exp\varphi. \tag{11c}$$

This system is the direct consequence of the equations (2) and constraints (3) for the defined variables $\varphi(\xi^0, \xi^1)$ and $\alpha_\pm(\xi^0, \xi^1)$. For the first time this PDE - system was considered in the work [12], where the new integrable field model was suggested in $2D$ space-time. The introduction of the function φ and the functions $\rho_\pm$ as the world-sheet parameters is justified by the following formulae for the first (**I**) and the second (**II**) forms of the world-sheet:

$$\mathbf{I} = -\frac{\varkappa^2}{2}\, \mathrm{e}^{-\varphi} d\xi_+ d\xi_- \,, \qquad \mathbf{II} = \varkappa[\rho_+ d\xi_+^2 - \rho_- d\xi_-^2] \,.$$

The standard method of geometrical description of a string [7] uses the equations (11a) and (11b) deduced from the Gauss and Peterson-Kodazzi

equations. In the standard approach the inequalities $\rho_\mp > 0$ are fulfilled. In this case the conformal transformations

$$\xi_\pm \longrightarrow \widetilde{\xi}_\pm = A_\pm(\xi_\pm)\,, \qquad A' \neq 0\,, \tag{12}$$

allow to reduce the equation (11a) to the Liouville equation; the form $\mathbf{I}$ will be the only fundamental geometrical object here. We are considering the enlarged class of the world-sheets for which real functions $\rho_\pm$ will be arbitrary differentiable functions. For example, the identity $\rho(\xi) \equiv 0$ should be fulfilled on any interval $[a, b] \subset [0, \pi]$. We must emphasize that in this case there are no transformations (12) that reduce the equation (11a) to the Liouville equation globally. The group G of the system (11) invariancy is much wider then the group (12). Indeed, let the functions $\varphi(\xi_+, \xi_-)$, $\rho_\pm(\xi_\pm)$ and $\alpha_\pm(\xi_+, \xi_-)$ be solutions for the system (11). Then the transformation

$$(\varphi, \rho_\pm, \alpha_\pm) \longrightarrow (\tilde{\varphi}, \tilde{\rho}_\pm, \tilde{\alpha}_\pm), \tag{13}$$

gives the new solution for the system (11) if

$$\begin{aligned}
\tilde{\varphi}(\xi_+, \xi_-) &= \varphi(A_+(\xi_+), A_-(\xi_-)) + f_+(\xi_+) + f_-(\xi_-),\\
\tilde{\rho}_\pm(\xi_\pm) &= \rho(A_\pm(\xi_\pm))A'_\pm(\xi_\pm)\exp\left(-f_\pm(\xi_\pm)\right),\\
\tilde{\alpha}_\pm(\xi_+, \xi_-) &= \alpha_\pm(A_+(\xi_+), A_-(\xi_-))\exp\left(f_\pm(\xi_\pm)\right) + g_\pm(\xi_\pm).
\end{aligned}$$

for arbitrary real functions $f_\pm(\xi)$, $g_\pm(\xi)$ and such real functions $A_\pm(\xi)$ where the conditions $A'_- A'_+ \neq 0$ are fulfilled. From the geometrical point of view, two kinds of the transformations (13) exist. The first kind corresponds to the conformal reparametrizations of the same world-sheet. The equalities

$$f_\pm(\xi) = -\ln A'_\pm(\xi) \tag{14}$$

extract these transformations from the group G. The second kind is all the other transformations which connect different world-sheets.

2 Factorization prosedure.

In this section we will be investigating the orbits of the group G. The results obtained here will help us to construct the anyon model – as certain quantum system that has exitations with arbitrary spin. This effect is due to the property of the group $SO(2)$ only; that is why both relativistic and non-relativistic models are interesting. Our consideration started with relativistic objects, but in the next section we are going to reduce our theory to non-relativistic case. The relativistic case will be considered in a separate work.

Now we continue the investigations of the local properties of the objects on interval $\xi^1 \in [0, \pi]$. The boundary conditions will be taken into account later. Let the vectors $\mathbf{b}_\mu \in E_{1,3}$ be constant vectors so that $\mathbf{b}_\mu \mathbf{b}_\nu = g_{\mu\nu}$. Let the vector - matrix $\hat{\mathbf{E}}_\mathbf{0} = \mathbf{b}_0 \mathbf{1_2} - \sum_{i=1}^{3} \mathbf{b}_i \boldsymbol{\sigma_i}$ correspond to the basis $\mathbf{b}_\mu$. It is clear that

$$\hat{\mathbf{E}}_\pm(\xi_\pm) = T_\pm(\xi_\pm) \hat{\mathbf{E}}_\mathbf{0} T_\pm^\top(\xi_\pm)\,, \tag{15}$$

where $T_\pm(\xi) \in SL(2,R)$. The equality

$$K(\xi^0, \xi^1) = T_+(\xi_+) T_-^{-1}(\xi_-) \tag{16}$$

is a sequence of the formula (8). Our next step is the reconstruction of the tangent vectors $\partial_\pm \mathbf{X}(\xi_\pm)$ through the matrix elements $t_{ij\pm}$ of the matrices $T_\pm$. Taking into account the formula (15) and the definition of the matrices $\hat{\mathbf{E}}_\pm(\xi_\pm)$, we obtain the following equalities:

$$\pm\,\partial_\pm \mathbf{X}(\xi_\pm) = \frac{\varkappa}{2}\Big[\left(t_{i1\pm}^2 + t_{i2\pm}^2\right)\,\mathbf{b}_0 - 2\,(\,t_{i1\pm} t_{i2\pm})\,\mathbf{b}_1 - \left(t_{i1\pm}^2 - t_{i2\pm}^2\right)\,\mathbf{b}_3\Big]\,, \tag{17}$$

where index i corresponds to the sign $\pm$ according to the rule $i = \frac{3\mp 1}{2}$. To reconstruct the whorld-sheet from the derivatives $\partial_\pm \mathbf{X}$ we must add the constant vector $\mathbf{Z}$.

The following proposition can be deduced directly from the definitions of the matrices $T_\pm$ and K:

Proposition 1 *The matrices $T_\pm$ are the solutions for the linear problems*

$$T_\pm'(\xi) + Q_\pm(\xi) T_\pm(\xi) = 0\,, \tag{18}$$

where

$$Q_-(\xi^0, \xi^1) = K^{-1}\partial_- K\,, \quad Q_+(\xi^0, \xi^1) = -(\partial_+ K) K^{-1}\,. \tag{19}$$

The global Lorenz transformations in our (3D) theory are the transformations

$$\hat{\mathbf{E}}_\mathbf{0} \longrightarrow \widetilde{\hat{\mathbf{E}}}_0 = \mathcal{B} \hat{\mathbf{E}}_\mathbf{0} \mathcal{B}^\top\,, \tag{20}$$

where the constant matrix $\mathcal{B} \in SL(2,R)$. It is clear that these transformations correspond to the arbitrariness for the matrix - solution of the systems (18):

$$T_\pm \longrightarrow \widetilde{T}_\pm = T_\pm \mathcal{B}^{-1}\,. \tag{21}$$

Thus the coefficients of the problems (18) are local functions of the introduced variables φ, $\rho_\pm$ and $\alpha_\pm$. These coefficients are relativistic invariants. For example, the equalities

$$Q_{12+} = -\rho_+\,, \qquad Q_{21-} = -\rho_-\,, \tag{22}$$

will be important for our subsequent considerations.

Let G_0 be the subgroup of the group G so that $A_\pm(\xi) \equiv \xi$ for all transformations (13). Then the following proposition is true:

Proposition 2 *If the group G_0 transforms the solution $\{\varphi,\ \rho_\pm,\ \alpha_\pm\}$ of the system (11), the matrices $T_\pm$ are transformed as follows:*

$$T_\pm \longrightarrow \tilde{T}_\pm = G_\pm^{-1} T_\pm \,, \tag{23}$$

where

$$G_+ = \begin{pmatrix} \mathrm{e}^{f_+/2} & 0 \\ g_+ \mathrm{e}^{-f_+/2} & \mathrm{e}^{-f_+/2} \end{pmatrix} , \qquad G_- = \begin{pmatrix} \mathrm{e}^{-f_-/2} & g_- \mathrm{e}^{-f_-/2} \\ 0 & \mathrm{e}^{f_-/2} \end{pmatrix} .$$

Proof. The proof is a direct consequence of the formulae (10), (16) and an explicit form for the transformations (13).

Let us take into account the boundary conditions for the field $\mathbf{X}(\xi^0, \xi^1)$. The standard analysis leads to equalities

$$\mathbf{e}_+(\xi) = \mathbf{e}_-(-\xi)\,, \qquad \mathbf{e}_+(\pi + \xi) = \mathbf{e}_-(\pi - \xi)\,. \tag{24}$$

These equalities mean that we can consider 2π-periodical vector field $\mathbf{e}(\xi) \equiv \mathbf{e}_+(\xi)$ which is defined for all real ξ instead of the fields $\mathbf{e}_+$ and $\mathbf{e}_-$ for $\xi \in [0, \pi]$. The function $\mathbf{e}_+$ (or the element $(\hat{\mathbf{E}}_+)_{11}$) and the function $\mathbf{e}_-$ (or the element $(\hat{\mathbf{E}}_-)_{22}$) are constrained by the conditions (24). We extend these constraints on all elements of the matrices $\hat{\mathbf{E}}_\pm(\xi)$. Thus the matrices $\hat{\mathbf{E}}_\pm(\xi)$ will be 2π-periodical matrices on real axis and $\hat{\mathbf{E}}_+(\xi) = K_0 \hat{\mathbf{E}}_-(-\xi) K_0^+$, where $K_0 = i\boldsymbol{\sigma}_\mathbf{2}$. Consequently, the equalities

$$T_-(\xi) = -K_0 T_+(-\xi)\,, \qquad T_+(\xi + 2\pi) = \pm T_+(\xi)\,, \tag{25}$$

will be true. Further we will be considering the matrix $T(\xi) \equiv T_+(\xi)$ only. It is clear that $T(\xi) = T_0(\xi) B$, where the constant matrix $B \in SL(2, R)$ and the matrix $T_0(\xi)$ satisfies the boundary condition $T_0(0) = I_2$. The elements of the matrix $T_0(\xi)$ will be single-valued functions from the coefficients Q_{ij} i.e. the functions φ and $\alpha_\pm$. Thus the parametrization (5) has been realized.

The formulae (16) and (25) allow us to continue the functions $\varphi(\cdot, \xi^1)$, $\alpha_\pm(\cdot, \xi^1)$ and $\rho_\pm(\xi^1)$ on all real axis. For example, $\rho_\pm(\xi_\pm) = \rho(\pm\xi_\pm)$, where $\rho(\xi)$ will be a 2π-periodical differentiable function.

Going back to the group (13), we can consider 2π-periodical function $f(\xi)$ instead of the functions $f_\pm$ which are connected by formulae $f_\pm(\xi_\pm) = f(\pm\xi_\pm)$, similar statement will be true for the functions $g_\pm(\xi)$. We can always demand that

$$\int\limits_0^{2\pi} f(\xi)d\xi = 0\,. \tag{26}$$

or redefine the constant $\varkappa$ if the eq. (26) is not true. Next, we must restrict the set of the functions $A_\pm(\xi)$ by the condition

$$A_\pm(\xi_\pm) = \pm A(\pm\xi_\pm)\,, \qquad A(\xi+2\pi) = A(\xi) + 2\pi\,,$$

where $A'(\xi) \neq 0$. Obviously, we can expand the action of the group G on 2π-periodical matrix $T(\xi)$. Let $\mathsf{G}_0[T]$ denote the orbit of the group G_0 for matrix $T(\xi)$. Then the following proposition will be fulfilled.

Proposition 3 *There exists the unique $SO(2)$ matrix $\mathcal{U} \in \mathsf{G}_0[T]$ solving the 2π-periodical linear problem*

$$\mathcal{U}'(\xi) + Q(\xi)\mathcal{U}(\xi) = 0\,, \tag{27}$$

where $Q(\xi) = -\rho(\xi)\boldsymbol{\sigma}_+ + \rho(\xi)\boldsymbol{\sigma}_-$.

Proof. Indeed, let us consider the Iwasawa decomposition for the matrix $T(\xi)$ such that $T = \mathcal{E}NU$ where the matrix $\mathcal{E}$ is a diagonal matrix with positive elements, $\mathcal{N}$ is a lower triangular matrix and $\mathcal{U} \in SO(2)$. The statement of Proposition 3 is a sequence of the unique existence of Iwasawa decomposition for any matrix $T \in SL(2,R)$, rule (23) for the matrix $T = T_+$ transformation and corresponing rule for the matrix Q. We use the same characters for matrices Q (the same concerns the coefficients ρ) both in the linear problem (18) and in the linear problem (27); we hope that these notations won't lead to any ambiguities.

The group G_0 can be decomposed into two kinds of the special transformations:

$$\alpha_\pm \;\rightarrow\; \alpha_\pm + g_\pm\,, \tag{28}$$

$$\varphi \;\rightarrow\; \varphi + f_+ + f_-\,, \quad \rho_\pm \rightarrow \rho_\pm e^{-f_\pm}\,, \quad \alpha_\pm \rightarrow \alpha_\pm e^{f_\pm}\,. \tag{29}$$

The following proposition is true.

Proposition 4 *The transformation (28) does not change the world-sheet; the transformation (29) transforms the world-sheet to the other world-sheet such that*

$$\mathbf{I} \;\longrightarrow\; \widetilde{\mathbf{I}} = \mathbf{I}\exp[-f_+ - f_-]\,, \tag{30}$$

$$\mathbf{II} \;\longrightarrow\; \widetilde{\mathbf{II}} = \varkappa[\rho_+ e^{-f_+} d\xi_+^2 - \rho_- e^{-f_-} d\xi_-^2]\,. \tag{31}$$

Proof. The proof is a sequense of the explicit formulae (17) for tangent vectors $\partial_\pm X(\xi_\pm)$, the explicit formulae for the forms $\mathbf{I}$ and $\mathbf{II}$ and rules (23) for transformations of the matrix elements $t_{ij\pm}$. Note that the existence of the transformations (28), which do not change the world-sheet, is the consequence of the arbitrariness in the definition of matrices $\hat{\mathbf{E}}_\pm(\xi_\pm)$.

Let us consider the set of world-sheets $\mathcal{X}$ introduced in the beginning of the paper. It will be recalled that space-time symmetry group here is the $3D$ Poincaré group $E(1,2)$. The object of our subsequent investigations is the factor-set $\mathcal{X}/\mathsf{G}_0$ only. Let us investigate the parametrization of the corresponding cosets. We are going to construct the parameters that can be separated into two sets. The first set will contain the finite number of "external" variables that parametrize certain space-time symmetry group in some way. The second set will be invariant under this group (the "internal" variables). For the set $\mathcal{X}$, for example, the "external" variables are the constant vector $\mathbf{Z} \in E_{1,2}$, the matrix $B \in SL(2,R)$ (see (5)) which parametrize the group $E(1,2)$ locally, and the quantity $\varkappa$.

Taking into account the Proposition 2.3 we can select the representatives in every coset so that $T_\pm = \mathcal{U}_\pm \in SO(2)$. Thus we have

$$\mathcal{U}(\xi) \equiv \mathcal{U}_+(\xi) = \mathcal{U}_0(\xi)U(\beta)\,, \qquad \mathcal{U}_0(0) = 1_2\,, \qquad \mathcal{U}(\beta) = \begin{pmatrix} \cos\beta & \sin\beta \\ -\sin\beta & \cos\beta \end{pmatrix}. \tag{32}$$

Because of the boundary conditions for the matrix $\mathcal{U}_0(\xi)$, the one-to-one correspondence $\rho(\xi) \leftrightarrow \mathcal{U}_0(\xi)$ exists. It is easy to see that

$$\mathcal{U}_0(\xi) = \begin{pmatrix} \cos I(\xi) & \sin I(\xi) \\ -\sin I(\xi) & \cos I(\xi) \end{pmatrix}, \qquad I(\xi) = \int_0^\xi \rho(\eta)d\eta\,. \tag{33}$$

In accordance with the second formulae (25) the matrix $\mathcal{U}_0(\xi)$ must be (anti)periodical. This fact means that the condition

$$\int_0^{2\pi} \rho(\eta)d\eta = \pi n\,, \qquad n = 0, \pm 1, \pm 2 \ldots . \tag{34}$$

must be fulfilled. Tangent vectors $\partial_\pm \mathbf{X}(\xi_\pm)$ are defined through 2π-periodical vector-function $\mathbf{e}(\xi)$ as follows:

$$\pm\, \partial_\pm \mathbf{X}(\xi_\pm) = \varkappa \mathbf{e}(\pm\xi_\pm)\,, \tag{35}$$

where

$$\mathbf{e}(\xi) = \frac{1}{2}\Big[\mathbf{b}_0 - \sin(2I(\xi)+2\beta)\mathbf{b}_1 - \cos(2I(\xi)+2\beta)\mathbf{b}_3\Big]\,.$$

It is clear that $X_0(\xi^0,\xi^1) = \varkappa\xi^0\mathbf{b}_0 + Z_0$ for our gauge. To reconstruct the spatial coordinates $X_j(\xi^0,\xi^1)$ $(j=1,3)$ of the world-sheet through the derivatives, we must introduce a two-dimensional vector with components Z_1 and Z_3. Thus we have the following one-to-one correspondence:

$$\Big(X_1(\xi^0,\xi^1),\, X_3(\xi^0,\xi^1)\Big) \longleftrightarrow \Big(\varkappa;\, Z_1, Z_3,\, \beta;\, \rho(\xi)\Big)\,. \tag{36}$$

The variables $\rho(\xi)$ and $\varkappa$ will be invariant under the group $E(2)\times\mathcal{T}_0$, where $E(2)$ is the group of the motions for the spatial plane $E_2 \perp \mathbf{b}_2$ and $\mathcal{T}_0$ is the group of time shifts. The variables (Z_1, Z_3, β) are transformed under space translations and space rotations in obvious manner.

Thus the following proposition will be true.

Proposition 5 *A space-time symmetry group for set $\mathcal{X}/\mathsf{G}_0$ will be the group $E(2)\times\mathcal{T}_0$.*

Where were the Lorentz boosts lost? It appears that two operations are non-commutative: the boost in the space-time $E_{1,2}$ and the selection of the gauge $T(\xi)\equiv\mathcal{U}(\xi)$. Thus the Lorentz boosts transform the functions $\rho(\xi)$ one through the other and will be "internal" transformations here.

In the context of the factorization procedure defined above, we can write the principal minor K_{11} of the matrix $K(\xi^0,\xi^1)$ as the function of the quantity ρ. To do it we must extract the element K_{11} from the formula (10). The result is as follows:

$$\exp[-\varphi(\xi^0,\xi^1)] = \sin^2\int_{-\xi_-}^{\xi_+}\rho(\eta)d\eta\,, \qquad \xi_\pm = \xi^1 \pm \xi^0\,. \tag{37}$$

This equality can be considered as the geometrical gauge condition for our theory. It must be emphasized that the arbitrariness (12) has not been fixed anywhere.

3 Dynamical system.

Let us write the formulae for Nöether invariants of the action (1):

$$P_\mu = \gamma \int_0^\pi \partial_0 X_\mu \, d\xi^1 \,, \qquad M_{\mu\nu} = \gamma \int_0^\pi (\partial_0 X_\mu X_\nu - \partial_0 X_\nu X_\mu)\, d\xi^1 \,.$$

The formulae (35) make it possible to calculate the components P_μ through the variables $\rho(\xi)$, β, and $\varkappa$; for example, the string energy $P_0 = \pi\gamma\varkappa$. As it has been proved above, space-time symmetry group of our system will be the group $E(2) \times \mathcal{T}_0$; and that is why we will use the formulae for Nöether invariants for spatial indices only. The following expressions can be deduced for the quantities $\mathbf{P}^2 = P_1^2 + P_3^2$ and $\mathsf{S} = M_{13} - Z_1 P_3 - Z_3 P_1$:

$$\mathbf{P}^2 = \pi^2\gamma^2\varkappa^2 F_P[\rho]\,, \tag{38}$$

$$\mathsf{S} = \frac{\pi\gamma\varkappa^2}{2} F_S[\rho]\,, \tag{39}$$

where

$$\begin{aligned} F_P[\rho] &= \frac{1}{4\pi^2} \int\limits_0^{2\pi}\int\limits_0^{2\pi} d\xi d\overline{\xi} \cos\Big(2\int_{\overline{\xi}}^{\xi} \rho(\eta)d\eta\Big)\,, \\ F_s[\rho] &= \frac{1}{2\pi} \int\limits_0^{2\pi}\int\limits_0^{2\pi} d\xi d\overline{\xi}\, W(\xi - \overline{\xi}) \sin\Big(2\int_{\overline{\xi}}^{\xi} \rho(\eta)d\eta\Big)\,, \\ W(\xi) &= \frac{1}{2\pi i}\sum_{n\neq 0} \frac{1}{n} e^{-in\xi} = \frac{\xi}{2\pi} - \left[\frac{\xi}{2\pi}\right] - \frac{1}{2}\,. \end{aligned}$$

Proposition 6 *The quantities P_1, P_3, S and $\rho(\xi)$ are constrained by the following condition:*

$$2\pi\gamma \mathsf{S} F_P[\rho] = \mathbf{P}^2 F_s[\rho]\,. \tag{40}$$

Proof. The proof is the exclusion of the variable $\varkappa$ from the formulae (38) and (39).

The dynamical system corresponding to the set $\mathcal{X}/\mathsf{G}_0$ is defined as follows.

- Space-time $E_{1,2}$ is reduced to "space and time" E_2 and R_1. The corresponding group of space-time symmetry is reduced to the group $E(2) \times \mathcal{T}_0$. Thus there are no world sheets (as geometrical objects) from this moment but the moving planar curves still exist.

- We define the constants (P_1, P_3, S) as the dynamical variables in our theory instead the variables β and $\varkappa \in (0, \infty)$. The corresponding set of the variables $(\rho(\xi)\,; P_1, P_3; B_1, B_3; \mathsf{S})$ we note as $\mathcal{W}$. Since we introduced three variables instead of two, the condition (40) must be imposed as the constraint; the symbol $\mathcal{V}$ denotes the corresponding surface. Then the following one-to-one correspondence is fulfilled:

$$\mathcal{X}/\mathsf{G}_0 \longleftrightarrow \mathcal{V} \subset \mathcal{W}\,. \tag{41}$$

- Supposing the variables (P_1, P_3, S) and ρ are independent we must close the domain $(0, \infty)$ for the constant $\varkappa$ by adding the boundary points $\varkappa = 0$ and $\varkappa = \infty$. Indeed, in accordance with our initial supposition the constant $\varkappa$ is a non-zero finite constant; for this domain the identity $\rho \equiv 0$ leads to the equalities $|\mathbf{P}| = 0$, $\mathsf{S} = 0$. Of couse there are no strings that correspond to the points $\varkappa = 0$ and $\varkappa = \infty$.

- We extend the group $E(2) \times \mathcal{T}_0$ to Galilei group $\mathcal{G}_2$. Indeed, the transformation

$$\mathbf{P} \to \widetilde{\mathbf{P}} = \mathbf{P} + c\mathbf{v}\,, \qquad \mathbf{v} = v_1\mathbf{b}_3 + v_3\mathbf{b}_3 \quad (c, v_j = const)$$

  defines Galilei boosts on the set of (independent) cordinates $(\rho(\xi^1 + \xi^0)\,; P_1, P_3; Z_1, Z_3; \mathsf{S})$ and is quite natural here.

- The central extension $\widetilde{\mathcal{G}}_2$ will be considered instead of $\mathcal{G}_2$; this step allows us to introduce an additional "in-put" parameter m_0 as a central charge and quantize theory [1].

- We use the variables

$$B_j = m_0\left(Z_j - \frac{\xi^0}{\gamma}\,P_j\right)\,, \qquad j = 1, 3$$

  instead of the variables Z_j.

After the reduction to the nonrelativistic case the following problem appears: what function will be the energy of constructed dynamical system? There are three Cazimir functions for central extended Galilei algebra:

$$\hat{C}_1 = m_0\hat{I}\,, \quad \hat{C}_2 = \left[\hat{M}_{13} - \hat{B}_1\hat{P}_3 - \hat{B}_3\hat{P}_1\right]^2, \quad \hat{C}_3 = \hat{H} - (1/2m_0)\hat{\mathbf{P}}^2\,,$$

where $\hat{I}$ is a unit operator, quantities $\hat{M}_{13}$, $\hat{H}$, $\hat{P}_i$ and $\hat{B}_i$ – generators of rotations, time and space translations and Galilei boosts correspondently. It

[1] We consider the one-parameter extension only.

is well-known that Cazimir function C_3 is interpreted as the internal energy of a "particle" (i.e. of our dynamical system). Thus the definition of the full energy as the function

$$\mathsf{E} = \frac{\mathbf{P}^2}{2m_0} + h[\rho]\,, \tag{42}$$

where $h[\rho]$ is the hamiltonian for "internal" variable $\rho(\xi^1 + \xi^0)$, will be quite natural.

Let us define the hamiltonian structure in our theory as follows:

- the phase space $\mathcal{H} = \overline{\mathcal{W}}$ with fundamental coordinates [2]

$$\Big(\rho(\xi)\,;\; P_1, P_3,\; B_1, B_3;\; \mathsf{S}\Big)\;;$$

- Poisson brackets

$$\begin{aligned} \{\rho(\xi), \rho(\eta)\} &= -\frac{1}{4}\,\delta'(\xi - \eta)\,, &(43)\\ \{P_i, B_j\} &= m_0\delta_{ij} &(44) \end{aligned}$$

  (other possible brackets equal zero);

- constraints (34) and (40) (the constraint surface will be the set $\overline{\mathcal{V}}$);

- hamiltonian

$$H = \frac{\mathbf{P}^2}{2m_0} + 2\int_0^{2\pi} \rho^{\,2}(\xi)\, d\xi + l(\xi^0)\Phi\,,$$

  where the function l is a lagrange multiplier.

The phase space will be as follows:

$$\mathcal{H} = \mathcal{H}_\rho \times \mathcal{H}_2\,,$$

where $\mathcal{H}_\rho$ is the phase space of internal degrees of freedom (it is parametrized by the function $\rho(\xi)$) and $\mathcal{H}_2$ is the phase space of a free particle on a plane E_2. The model is non-trivial because of the constraint (40) that entangles the internal and external variables. Topological constraint (34) selects the symplectic sheets in the space $\mathcal{H}_\rho$.

[2] closure in the weak topology that is defined by the function $\varkappa = \varkappa[\rho(\xi); \mathsf{S}]$

4 Quantization

The theory considered above leads to the following natural structure of the Hilbert space of the quantum states:

$$\mathbf{H} = \int_S \mathbf{H}_S\,, \qquad \mathbf{H}_S = \mathbf{H}_2 \times \mathbf{H}_\psi\,.$$

The space $\mathbf{H}_2 \approx L^2(\mathsf{R}_2)$ is the Hilbert space for a free non-relativistic particle with internal moment S on a plane; the space $\mathbf{H}_\psi$ is the Fock space of the "internal degrees of freedom" $\rho(\xi)$. The constraint (40) leads to the equation for physical states $|\psi_s\rangle \in \mathbf{H}_S$:

$$\Big(\gamma \mathsf{S} \hat{I}_2 \otimes \widehat{F}_P - \widehat{\boldsymbol{P}}^2 \otimes \widehat{F}_S\Big)|\psi_s\rangle = 0\,. \tag{45}$$

The stationary Schrödinger equation

$$\widehat{H}|\psi_s\rangle \equiv \Big((1/2m_0)\widehat{\boldsymbol{P}}^2 \otimes \hat{I}_\psi + \hat{I}_2 \otimes \hat{h}\Big)|\psi_s\rangle = \mathsf{E}|\psi_s\rangle \tag{46}$$

defines the energy of our system together with the equation (45). The following notations are used here: symbol $\widehat{\ldots}$ denotes the quantized functions in the corresponding space so that $\hat{I}_i$ is the unit operator in the space $\mathbf{H}_i$ $(i = 2, \psi)$ and so on. The states $|\psi_s\rangle \in \mathbf{H}_S$ that solve the system (45) – (46) can be considered as entangled states such that

$$|\psi_s\rangle = \sum_n a_n |f_{s,n}\rangle |\alpha_n\rangle\,,$$

where $|f_{s,n}\rangle \in \mathbf{H}_2$ and $|\,\alpha_n\rangle \in \mathbf{H}_\psi$. Let the space $\mathbf{H}_2$ be the (framed) space $L^2(\mathsf{R}_2)$ so that $\widehat{\boldsymbol{P}}^2 = -\Delta_2$. Suppose that the wave functions $\langle \mathbf{Z}|f_{s,n}\rangle$ have a form

$$\langle \mathbf{Z}|f_{s,n}\rangle = J_{l-s}(k_{s,n} r) e^{i(l-s)\phi}\,, \qquad \mathbf{Z}:\,(Z_1 = r\cos\phi,\; Z_3 = r\sin\phi)\,,$$

where the functions J_ν are Bessel functions and number l is a total moment of the whole system. Then the real quantities $k = k_{s,n}$ and vectors $|\,\alpha\rangle = |\,\alpha_n\rangle$ are found from the following spectral problem in the space $\mathbf{H}_\psi$:

$$\Big(2\pi\gamma\,\mathsf{S}\widehat{F}_P - k^2 \widehat{F}_S\Big)|\,\alpha\rangle \;=\; 0\,, \tag{47}$$

$$\Big(\frac{k^2}{2m_0} + \hat{h}\Big)|\,\alpha\rangle \;=\; \mathsf{E}|\,\alpha\rangle\,. \tag{48}$$

The suggested scheme will be formal unless we quantize the function ρ, define the Hylbert space $\mathbf{H}_\psi$ and construct the corresponding operators $\widehat{F}_P$, $\widehat{F}_S$ and $\hat{h}$. To do this we apply the method of the boson-fermion correspondence. We intend to follow the work [13] where both the rigorous investigation and the detailed historical review of this procedure were carried out.

Let us define the fermionic field $\psi(\xi^0,\xi^1)=\psi(\xi^0+\xi^1)$ where 2π-periodical operator-valued function $\psi(\xi)$ is defined as follows:

$$\psi(\xi)=\sum_{n=1}^{\infty}a_n^* e^{-in\xi}+\sum_{n=0}^{\infty}b_n e^{in\xi}\,.$$

The fermionic operators a_n^* and b_n^* will be creation operators in the Fock space $\mathbf{H}_\psi$ with vacuum vector $|\,0\rangle$; the operators a_n and b_n will be the corresponding annihilation operators. Canonical anticommutation relations

$$[a_n^*,a_m]_+=\delta_{nm}\quad(n,m=1,2,\dots)\,,\qquad [b_n^*,b_m]_+=\delta_{nm}\quad(n,m=0,1,2,\dots)$$

are carried out. As the next step we consider the current $v(\xi)=:\psi^*(\xi)\psi(\xi):$ where the symbol $:\ :$ denotes the fermion ordering:

$$:\psi^*(\xi)\psi(\xi):=\lim_{\eta\to\xi}\Big(\psi^*(\eta)\psi(\xi)-\langle 0|\psi^*(\eta)\psi(\xi)|\,0\rangle\Big)\,.$$

The current $v(\xi)$ will be the well-defined bozonic field with commutation relations:

$$[v(\xi),v(\eta)]=i\delta'(\xi-\eta)\,. \tag{49}$$

The charge

$$\Lambda=\frac{1}{2\pi}\int_0^{2\pi}v(\xi)d\xi=\sum_{n=0}^{\infty}b_n^*b_n-\sum_{n=1}^{\infty}a_n^*a_n$$

has integer eigenvalues. We can decompose the space $\mathbf{H}_\psi$ as follows

$$\mathbf{H}_\psi=\bigoplus_{n=-\infty}^{\infty}\mathbf{H}_n\,,$$

where space $\mathbf{H}_n$ is the eigenspace corresponding to the eigenvalue n of operator Λ. Details can be found in the work [13].

Here is the quantization postulate for the internal degrees of freedom – function $\rho(\xi)$:

$$\rho(\xi)\longrightarrow\hat{\rho}(\xi)\equiv\frac{1}{2}\,v(\xi)\,.$$

What is the motivation for this postulate here? Don't take into account the boundary conditions (4), let us return to the linear problems (18) and consider the functions $t_{ij\pm}(\xi_\pm)$. The following proposition will be true.

Proposition 7 *The objects* $\begin{pmatrix} t_{i1\pm} \\ t_{i2\pm} \end{pmatrix}$ *will be the Majorana spinors in the space-time* $E_{1,2}$ *for every sign* $\pm$ *and* $i = 1, 2$*; the objects* $\begin{pmatrix} t_{ij+} \\ t_{ij-} \end{pmatrix}$ *will be the spinors in tangent plane* $E_{1,1}$ *for every* $i, j = 1, 2$.

Proof. The first statement follows from the formulae (20) and (21). To prove the second statement let us fulfill the Lorentz transformation for tangent plane $E_{1,1}$: $\xi_\pm \to \tilde{\xi}_\pm = \lambda^{\pm 1}\xi_\pm$. The formulae (17) demonstrate that the quantities $t_{ij\pm}$ are transformed as $t_{ij\pm} \to \tilde{t}_{ij\pm} = \lambda^{\pm 1/2} t_{ij\pm}$. Thus the objects $\begin{pmatrix} t_{ij+} \\ t_{ij-} \end{pmatrix}$ are transformed as spinors in the "space-time" $E_{1,1}$.

The elements of the matrices $Q_\pm$ are the bilinear combinations of the objects $t_{ij\pm}$ and $t'_{ij\pm}$. Thus the interpretation of the quantum function ρ (it defines the elements of the matrices $Q_\pm$ as considered above) as the current of free $2D$ fermionic field will be quite natural in our model.

In accordance with definition of the charge Λ, the topological constraint (34) is fulfilled identically for our quantization. Let us construct the operators $\widehat{F}_P$, $\widehat{F}_S$. To do this we use[3] the Theorem 6.1 from the work [13]:

$$\Big[\exp\Big(-i(\xi-\overline{\xi})\Big)-1\Big] : \psi^*(\xi)\psi(\overline{\xi}) := \vdots \exp\left(i\int_{\overline{\xi}}^{\xi} v(\eta)d\eta\right) \vdots - 1\,, \qquad (50)$$

where symbol $\vdots \ \vdots$ denotes the boson ordering. Taking into account formula (50) and the classical formulae for the quantities F_P and F_S, we find the explicit formulae for the operators $\widehat{F}_P$ and $\widehat{F}_S$ in the space $\mathbf{H}_\psi$:

$$\begin{aligned}
\widehat{F}_P &= \mathrm{Re}\frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi} d\xi d\overline{\xi}\,\Big[1+\Big(\exp\Big(-i(\xi-\overline{\xi})\Big)-1\Big) : \psi^*(\xi)\psi(\overline{\xi}) : \Big] = \\
&= 1 - b_0^* b_0 - a_1^* a_1\,, \qquad (51) \\
\widehat{F}_S &= \mathrm{Im}\frac{1}{2\pi}\int_0^{2\pi}\int_0^{2\pi} d\xi d\overline{\xi}\, W(\xi-\overline{\xi})\Big(\exp\Big(-i(\xi-\overline{\xi})\Big)-1\Big) : \psi^*(\xi)\psi(\overline{\xi}) : = \\
&= b_0^* b_0 - a_1^* a_1 + \sum_{k=1}^{\infty}\frac{a_{k+1}^* a_{k+1} - b_k^* b_k}{k(k+1)}\,, \qquad (52)
\end{aligned}$$

These simple formulae justify our approach to the quantization of bosonic field $\rho(\xi)$. The following proposition is fulfilled:

[3]after the modification for the periodical case and chirality "+"

Proposition 8 *The eigenvalues of operator* $\widehat{F}_P$ *are the integer numbers* -1, 0, 1. *The eigenvalues of operator* $\widehat{F}_S$ *form the everywhere dence set on interval* $[-2, 2]$.

Proof. The first statement is obvious. To prove the second statement let us note that $\sum_{n=1}^{\infty} 1/n(n+1) = 1$. It is clear that we can approximate every number $\beta \in (0,1)$ by sum $\sum_{n=1}^{\infty} \epsilon_n/n(n+1)$, where the factor ϵ_n can be 0 or 1. We omit the detailed algorithm here.

Let us select the physical states. It is clear that the considered object does not interact with anything. That is why we must exclude any states that lead to equality $k = 0$ or inequality $k^2 < 0$ in the system (47) - (48). Thus we must consider the states

$$b^*_{n_1} \ldots b^*_{n_k} a^*_{m_1} \ldots a^*_{m_l} |\, 0\rangle \qquad n_i \neq n_j\,, \quad n_j \neq 0\,, \qquad m_i \neq m_j\,, \quad m_j \neq 1\,.$$

These states correspond to the anyon with the arbitrary spin S that is connected with the energy by means of the formula

$$\mathsf{S} = \frac{m_0}{\pi\gamma}\Bigl(\mathsf{E} - \sum_{i=1}^{k} n_i - \sum_{i=1}^{l} m_i\Bigr)\left(\sum_{i=1}^{l} \frac{1}{m_i(m_i+1)} - \sum_{i=1}^{k} \frac{1}{n_i(n_i+1)}\right)\,. \tag{53}$$

We have $\langle\psi_{S_1}|\psi_{S_2}\rangle \propto \delta(S_1 - S_2)$ for the considered states; thus $|\psi_S\rangle \in \mathbf{H}'_S$ where the symbol $'$ denotes the framed Hylbert space. The formula (53) corresponds the case $k^2 > 0$ for $\mathsf{E} > \sum_{i=1}^{k} n_i + \sum_{i=1}^{l} m_i$ only.

5 Concluding remarks

We have constructed here the new dynamical system on a plane. The phase space of the constructed dynamical system has a "string sector" – the set $\mathcal{V}$ which is everywhere dence on the constraint surface $\overline{\mathcal{V}}$; this set corresponds bijectively to the theory of open string on a plane. This fact allows us to interpret this dynamical system as the extended particle. Obviously the space $\mathbf{H}_\psi$ is redundant for quantization of the field $\rho(\xi)$. Indeed, this space was constructed as the Fock space for fermionic field $\psi(\xi)$; the current $v(\xi)$ will be invariant for the transformations

$$\psi(\xi) \longrightarrow \widetilde{\psi}(\xi) = \psi(\xi)\exp[i\chi(\xi)]\,. \tag{54}$$

In our opinion, this problem can be solved in two ways. The first way is to pass from the space $\mathbf{H}_\psi$ to bosonic Hilbert space $\mathbf{H}_B$ which is connected with the space $\mathbf{H}_\psi$ by the formula $\mathbf{H}_\psi = \mathbf{H}_B \times \mathbf{H}_0$. The space $\mathbf{H}_0$ will be

the space for "zero mode" operator Λ and the operator p that is canonically conjugated with the Λ. Details can be found in the work [13]. The second way is to interpret the superfluous degrees of freedom. So, the "string sector" corresponds to the factor-set $\mathcal{X}/\mathsf{G}_0$; the superfluous degrees of freedom can be used, for example, to quantize the orbits of group G_0. This possibility will be investigated in subsequent works.

In this article we did not set ourselves any discussing of critical dimensions in string theory as an object. From the viewpoint of our approach, this question was discussed, for example, in the work [14], where the relativistic theory of the spinning string in four-dimensional space - time was suggested.

References

[1] F. Wilczek, *Phys. Rev. Lett.* **48**, 1144 (1982).

[2] R.B. Laughlin, *Phys. Rev. Lett.* **50**, 1395 (1983).

[3] M. Levin, X.-G. Wen, *Rev. Mod. Phys.* **77**, 871 (2005).

[4] B. Zwiebach, *A First course in String Theory.* (Cambr. Univ. Press, 2004).

[5] M. Green, J. Shwarz and E. Witten, *Superstring theory.* v.1. (Cambrige Univ. Press, 1987).

[6] S.V. Talalov, *Theor. Math. Phys.* **123**, 446 (2000), *Journ. of Phys. A* **22**, 2275 (1989).

[7] B.M. Barbashov and V.V. Nesterenko, *Introduction to the Relativistic String Theory.* (World Scientific, Teaneck, N.J. 1990).

[8] M.R. Anderson. *The mathematical theory of cosmic strings.* (IOP Publishing, 2003).

[9] G. Marmo, G. Scholarici, A. Simoni, F. Ventrigla, *Theor. Math. Phys.* (**144**, 1190 (2005).

[10] P.A.M. Dirac. Generalized hamiltonian dynamics. *Canadian Journ. of Math.* **2**, No 3, 129 (1950).

[11] S.V. Talalov, *Theor. Math. Phys.* (**152**), 1234 (2007).

[12] A. K. Pogrebkov, S.V. Talalov, *Theor. Math. Phys.* **70**, 241 (1987).

[13] A. K. Pogrebkov, *Russian Mathematical Surveys* **58**, No 5, 1003 (2003).

[14] S.V. Talalov, *Journ. of Phys. A.*, **32**, 845 (1999).